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Qualitative and numerical observations into the career trajectories of early career academics: A mathematical model approach

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Abstract

This study develops and analyzes a mathematical model that describes the interplay among different researcher groups within academia by focusing on how variables such as mentorship, access to funding, and academic peer influence affect the career trajectories of early-career, experienced, redundant, and successful researchers. Using stability theory and relevant theorems of differential equations, this study ensures the existence, uniqueness, positivity, and boundedness of the model's solutions. Additionally, the model's equilibrium states, are analyzed to determine the local asymptotic stability. Numerical simulations using the Runge-Kutta fourth-order (RK4) method, implemented in Python, are performed to illustrate issues related to career growth within the academic community. The model simulations results indicate that the long-term sustainability of academic progression is governed by the dynamic balance among mentorship effectiveness, funding allocation efficiency, and institutional support intensity. Qualitatively, this corresponds to maintaining some model interaction parameters $\beta_1, \beta_3, \beta_4$ (representing mentorship and funding influence rates) above their respective threshold values, while ensuring that the detrimental influence parameters η_1, η_2 and rate at which experienced researchers leave the academic δ_o, remain sufficiently small to prevent destabilization of the equilibrium states. The simulations further reveal that an increase in the positive interaction coefficients enhances the asymptotic stability of the equilibrium, thereby promoting persistence of productive researcher populations over time. On the other hand, excessive negative interactions induce a shift toward instability and decline in academic performance. Hence, the model provides a quantitative framework through which policymakers and academic institutions can identify critical thresholds and parameter regimes that guarantee resilience, equitable opportunity, and long-term systemic stability within the academic ecosystem.

Introduction

In the ever-evolving landscape of academia, the trajectory of a researcher's career is influenced by a myriad of factors ranging from institutional support to personal and professional interactions. Understanding these dynamics is important for academic institutions that aim to foster environments conducive to research innovation and academic success (Sherif et al. 2020).

Academic career progression is traditionally viewed as a linear pathway from early career stages to experienced and then to potentially successful positions. However, this progression is often non-linear and affected by various internal and external factors. Early career researchers (ECRs) face numerous challenges, including securing funding, navigating the thorough landscape of academic publishing, and establishing themselves in competitive research environments, (Bridle et al. 2013). Experienced researchers, while more established, continue to deal with pressures of maintaining their research relevance and funding, whereas successful researchers must balance research and increasingly demanding administrative or mentoring roles (Mason and Merga 2022).

Alternately, researchers who fail to secure adequate support or funding, or who are unable to effectively navigate the academic system, may find themselves in a redundant or poor category, struggling with career stagnation (Hughes 2012).

The dynamics of academic career progression have been the focus of extensive scholarly attention, primarily driven by the increasing competitiveness within higher education institutions globally. This study synthesizes important findings from three interconnected areas relevant to this study: which are, the impact of mentorship on academic careers, because mentorship has the potential to greatly enhance career trajectories, its actual impact depends heavily on the quality of the mentor-mentee relationship (Boeren et al. 2015).

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Funding is another critical component of academic career development. The availability of research funding directly affects the scope and ambition of academic projects, with well-funded researchers more likely to pursue innovative and high-risk projects. Moreover, funding security often influences job satisfaction and career longevity in academia. Alternately, the competitive nature of grant writing can have adverse effects. The pressure to secure funding can lead to research conservatism, where researchers favor safe topics over innovative or risky ones that could potentially lead to more significant breakthroughs. This behavior can stifle creativity and slow the pace of scientific advancement (Khoo *et al*. 2023).

The interactions among different researcher groups within academic institutions, namely early career researchers, experienced researchers, and successful researchers, play a significant role in shaping the academic environment due to collaborative networks in enhancing research output and career advancement (Sutherland 2017). However, the presence of negative dynamics, such as internal competition and lack of support, can lead to detrimental outcomes for certain groups, particularly early career and redundant researchers. Institutional policies also critically impact researcher trajectories (Sørensen and Kristensen 2022).

Policies concerning recruitment, tenure, and promotion directly influence the academic career ladder, often dictating the pace and direction of career progression. The effectiveness of these policies in promoting a fair and inclusive academic environment has been questioned, with some studies suggesting that they may inadvertently favor certain demographics or research disciplines over others, (Laudel and Gläser 2008).

Mathematical modeling has been a crucial tool in analyzing physical, biological and social phenomena. Not much work has been done in using mathematical modeling to analyze academic issues and their attendant psychosocial effects, but the few ones are presented below. The utilization of social platforms has significantly increased during the COVID-19 pandemic for both academic and non-academic purposes, raising concerns about the potential for social media addiction among students. (Bansal *et al*. 2000) studied this phenomenon and proposed a fractional-order mathematical model to analyze the impacts of social media on academic performance. Their study identified two equilibrium states namely, a social web-free equilibrium and an endemic equilibrium. They found that the social web-free equilibrium is globally asymptotically stable when the threshold value is below one, while endemic equilibrium points emerge when the threshold exceeds this value. Their work highlights that social media addiction can be mitigated by reducing the order of the derivative, showcasing the efficacy of fractional-order models over traditional integer-order models, suggesting the need for techniques to control social media consumption considering its profound influence on students' academic lives.

Also, (Mazzoleni *et al.* 2021) addressed the complexities of interdisciplinary interactions within academia, influenced by inbreeding-lobbying and institutional policies. The study introduced a mathematical model consisting of four coupled nonlinear ordinary differential equations to simulate the interactions between various academic behaviors and the advancement of knowledge, which is directly tied to disciplinary diversity. Their analysis through numerical bifurcation revealed complex behaviors such as multi stability and sustained oscillations, which are critical for understanding the barriers to interdisciplinary. The model also examined the effects of policies aimed at curbing inbreeding-lobbying practices by showing how cognitive rigidity fostered by entrenched academic lobbying can impede the flourishing of interdisciplinary environments crucial for innovation.

In light of the above studies, our work introduces a new dimension to the challenges faced by researchers at different career stages through mathematical modeling. Early career researchers (ECRs) often encounter significant hurdles in securing funding, navigating the complex landscape of academic publishing, and establishing a foothold in competitive research environments. More established researchers continue to struggle with maintaining research relevance and securing necessary funding. On the other hand, successful researchers juggle research activities with demanding administrative or mentoring roles. Alternatively, researchers who fail to secure adequate support or effectively navigate academic systems may become redundant, facing career stagnation.

To address these challenges, our work formulates a deterministic mathematical model that uses realistic variables and parameters to describe these pervasive academic issues and propose solutions. This model aims to provide a comprehensive framework that shows the realities of academic careers, thereby contributing valuable insights into academic career trajectories and the effectiveness of various support mechanisms.

Model Formulation and Analysis

The provided system of first order nonlinear ordinary differential equations models the dynamic interplay between different categories of researchers within an academic setting. The model includes compartments for academia who are Early Career Researchers $E_c(t)$, Experienced Researchers $E_x(t)$, Redundant or Poor Researchers $R_e(t)$, and Successful Researchers $S_c(t)$, so that the total host population of researchers $N_a(t)$, is such that $N_a(t) = E_c(t), +E_x(t) + R_e(t) + S_c(t)$ and the non-human compartment of funding is represented by F_d(t). Before the derivation of the model equations, some definitions and assumptions were made with the following:

Early Career Researchers (ECRs) E_c

- Description: This group includes postdoctoral fellows, junior faculty, and researchers who have recently completed their PhDs.
- Key Activities: Development of independent research agendas, seeking grant funding, publishing initial research, and beginning to take on teaching responsibilities.

Experienced Researchers E_x

- Description: Retired or Senior faculty researchers who have established a significant body of work and hold positions of influence.

- Key Activities: Advanced research projects, leadership in grant applications, mentorship of younger academics, and administrative roles.

Redundant/Poor Researchers R_e

- Description: Academics facing challenges in securing funding, publishing, or achieving tenure, regardless of their career stage.
- Key Activities: Improvement programs, collaboration opportunities, and additional support mechanisms to enhance research output and success.

Successful Researchers S_c

- Description: Academics who have demonstrated high levels of success in terms of funding, publications, and recognition.
- Key Activities: Leading major projects, mentoring, participating in high-level committees, and contributing to policy making in academia.

In view of the explanations above, the model equations are derived with explanations below.

The Early Career Researchers (ECRs) Dynamics (E_c)

New entrants join the ECRs pool (new hires, newly graduated PhDs, incoming postdocs) at rate Π, because universities and research institutes constantly recruit early-career academics; Π models this influx. With the quantity $\rho_1 E_c$, a fraction of ECRs mature into experienced researchers per unit time (obtaining tenure, promotion, or accumulating experience). Also, time and professional progress move some ECRs into the senior/experienced class.

Losses due to negative peer influence or discouragement occurs with the quantity $\frac{\beta_2 E_c R_e}{N_a}$ because contacts with redundant/stagnated researchers can demotivate ECRs by reducing productivity or causing career decline; those effects push ECRs toward the R_e pool. Here, frequency-dependent contact captures the idea that the more redundant researchers relative to the community, the higher the chance an ECR experiences negative influence. The quantity $\frac{\beta_1 E_c E_x}{N_a}$ denotes the interaction with experienced researchers which produce positive outcomes (mentoring leading to successful careers) while net loss term represents the total ECRs who, through these interactions, leave the pure ECRs class to become either successful, redundant, or experienced. ECRs, experienced interactions are a primary driver of career trajectories.

With the quantity $\frac{\beta_5 E_c E_x}{N_a}$, ECRs influenced by successful researchers may either escalate quickly (become successful or directly become experienced via recognition/collaboration) or be discouraged by high standards; again, the interaction leads to transitions out of E_c. A split parameter $\sigma_1$will determine whether the outcome tends toward E_x or S_c because successful researchers often recruit/mentor ECRs.

The funding-mediated transitions (Holling II) is denoted by the quantity $\frac{\beta_4 E_c F_d}{1+\alpha F_d}$ where access to funding helps ECRs scale projects, employ PhD students, and publish; this term model's conversion of ECR effort into career transitions driven by funding. The Holling II factor $\frac{F_d}{1+\alpha F_d}$ captures diminishing marginal returns of funding. Some small amounts of funding can have large marginal effects; beyond a point, extra funding has saturated impact. This prevents unrealistic unlimited benefits from arbitrarily large F_d, while μE_c denotes natural exits (leaving academia, career change, death). Therefore, the compartment model becomes

$$\frac{dE_c}{dt} = \Pi - \frac{\beta_1 E_c E_x}{N_a} - \frac{\beta_2 E_c R_e}{N_a} - \frac{\beta_4 E_c F_d}{1+\alpha F_d} - \frac{\beta_5 E_c S_c}{N_a} - (\mu + \rho_1) E_c \quad (1)$$

The Experienced Researchers Dynamics (E_a)

For the experienced researcher compartment E_a, ECRs who complete probation/tenure become experienced at the rate $\rho_1 E_c$. Also, at the rate $\sigma_1 \frac{\beta_5 E_c E_x}{N_a}$, a portion of ECR successful interactions result in ECRs moving into the experienced class (for example, gaining a faculty appointment or stable position before becoming successful"). Here σ_1 , σ_2 [0, 1] partitions the ECR and S_c interaction outcomes. Not all strong interactions create immediate success; some simply accelerate advancement to an experienced role.

Loss due to redeployment toward success (or reallocation) is given by $\sigma_2 \frac{\beta_3 E_x S_c}{N_a}$, while experienced researchers interacting with highly successful peers may be drawn into large collaborative projects where they functionally move into a different role (counted as growth in S_c rather than E_x, or conversely be overshadowed and shift roles). σ_2 partitions the $E_x S_c$ interaction, because experienced researchers sometimes transition into the successful category as they accumulate major grants / recognition.

Negative influence from redundancy occurs via the term $\frac{\beta_6 E_x R_e}{N_a}$ because experienced researchers may be negatively affected by association with stagnated colleagues (administrative load, demoralization) and so transition to R_e or exit because cross-influence exists at all career stages. Furthermore, retirement occurs at δ_o and natural death μ because, senior researchers retire or leave the active pool. Thus, the compartmental model equation becomes

$$\frac{dE_x}{dt} = \rho_1 E_c + \sigma_1 \frac{\beta_5 E_c E_x}{N_a} - \sigma_2 \frac{\beta_3 E_x S_c}{N_a} - \frac{\beta_6 E_x R_e}{N_a} - (\delta_o + \mu) E_a \quad (2)$$

For the Redundant/Poor Researchers Dynamics (R_e):

Entry via ECR E_c *and* R_e interactions is denoted by quantity $\frac{\beta_2 E_c R_e}{N_a}$. Peer effects where redundant researchers influence ECRs to reduce productivity or leave the positive trajectory. Cultural or methodological contagion of low productivity.

Entry via a fraction of ECR experienced interactions that produce negative outcomes: $\frac{\eta_1\beta_1E_cE_x}{N_a}$. Some mentorship interactions are harmful (mismatch, exploitation, gatekeeping) and cause ECRs to become stagnated; (η_1 $\eta_1 \epsilon$ [0, 1]) is the fraction leading to negative outcomes, because not all interactions with experienced researchers are beneficial.

Entry via a fraction of funding interactions that misfire is given by $\frac{\eta_2\beta_4E_cF_d}{1+\alpha F_d}$. Funding can also create pressure, mismanagement, or inequality that pushes some ECRs into poor outcomes, a fraction η_2 of funding-mediated transitions lead to redundancy. Funding is double-edged; bad distribution or pressure can harm careers.

The removal term is given by $(\delta_1+\mu)R_e$ which irrecoverable exits (leave academia, retirement, career change). Some redundant researchers exit the system or finally leave academic employment.

Thus, the model equation is given by

$$\frac{dR_e}{dt}=\frac{\eta_1\beta_1E_cE_x}{N_a}+\frac{\eta_2\beta_4E_cF_d}{1+\alpha F_d}+\frac{\beta_2E_cR_e}{N_a}-(\delta_1+\mu)R_e. \tag{3}$$

For the Successful Researchers Dynamics (S_c):

With the quantity, $\frac{(1-\eta_1)\beta_1E_cE_a}{N_a}$, the proportion $(1-\eta_1)$ of the ECR experienced interaction produces positive outcomes (e.g., good mentorship, co-authorship, promotion) that create successful researchers because many ECRs become successful through good mentoring and collaboration.

Also, with quantity $\frac{(1-\eta_2)\beta_4E_cF_d}{1+\alpha F_d}$, when funding is well used, it directly supports successful outcomes for ECRs. The proportion $(1-\eta_2)$ denotes the proportion of funding-driven transitions that produce success with funding enables research that leads to success, but not always.

Entry via experienced successful interactions occurs with the quantity $\frac{(1-\sigma_2)\beta_3E_xS_c}{N_a}$, because mentoring and co-investigations with existing successful researchers lift experienced researchers into the successful category (e.g., through collaborative PI roles). Existing successful researchers help expand the successful cohort via collaboration and visibility.

Entry via ECR successful interactions leads to success via $\frac{(1-\sigma_1)\beta_5E_cS_c}{N_a}$. The proportion $(1-\sigma_1)$ of ECR with S_c contact directly yields success (fast-track to PI roles, high-impact collaborations) because some relationships produce immediate success for ECRs.

With the term μS_c, successful researchers also leave the active pool due to retirement, career change, or death. Therefore, the model compartment is given by:

$$\frac{dS_c}{dt}=\frac{(\eta_1)\beta_1E_cE_x}{N_a}+\frac{(\eta_2)\beta_4E_cF_d}{1+\alpha F_d}-\frac{(1-\sigma_2)\beta_3E_aS_c}{N_a}-\frac{(1-\sigma_1)\beta_5E_cS_c}{N_a}-\mu S_c \tag{4}$$

For the funding dynamics (F_d):

We used the logistic function by $r\left(1-\frac{F_d}{\kappa}\right)F_d$. Funding availability can grow (e. g., new institutional initiatives, philanthropic growth) but is bounded by a carrying capacity κ, but are ultimately constrained by natural and systemic limits (e.g., finite budget allocations, competitive grant caps).

For the baseline depletion or consumption term $-\delta_3F_d$, funding is continuously spent and consumed or expires, because grants are time-limited and are spent on projects, while the funds are reduced by different classes of researchers at the quantity $(\theta_1E_c+\theta_2E_x+\theta_3E_c+R_e)F_d$, where $0<\theta_1<\theta_2<\theta_3<1$ denotes the modification factor representing funding attractions relative to each other. Therefore,

$$\frac{dF_d}{dt}=r\left(1-\frac{F_d}{\kappa}\right)F_d-\delta_3F_d+\alpha_sS_c \tag{5}$$

Coupling the governing dynamical equations in (1) – (5) becomes:

$$\left.\begin{aligned}
\frac{dE_c}{dt}&=\Pi-\frac{\beta_1E_cE_x}{N_a}-\frac{\beta_2E_cR_e}{N_a}-\frac{\beta_4E_cF_d}{1+\alpha F_d}-\frac{\beta_5E_cS_c}{N_a}-(\mu+\rho_1)E_c,\\
\frac{dE_x}{dt}&=\rho_1E_c+\sigma_1\frac{\beta_5E_cE_x}{N_a}-\sigma_2\frac{\beta_3E_xS_c}{N_a}-\frac{\beta_6E_aR_e}{N_a}-(\delta_o+\mu)E_x,\\
\frac{dR_e}{dt}&=\frac{\beta_2E_cR_e}{N_a}+\frac{(1-\eta_1)\beta_1E_cE_x}{N_a}+\frac{(1-\eta_2)\beta_4E_cF_d}{N_a}-(\delta_1+\mu)R_e,\\
\frac{dS_c}{dt}&=\frac{(\eta_1)\beta_1E_c}{N_a}+\frac{(\eta_2)\beta_4E_cF_d}{1+\alpha F_d}-\frac{(1-\sigma_2)\beta_3E_xS_c}{N_a}-\frac{(1-\sigma_1)\beta_5E_cS_c}{N_a}-\mu S_c,\\
\frac{dF_d}{dt}&=r\left(1-\frac{F_d}{\kappa}\right)F_d-\delta_3F_d-(\theta_1E_c+\theta_2E_x+\theta_3E_c+R_e)F_d
\end{aligned}\right\} \tag{6}$$

Subject to initial conditions:

$$E_c(0)\geq 0, E_x(0)\geq 0, R_e(0)\geq 0, S_c(0)\geq 0, F_d\geq (0) \tag{7}$$

Existence and Uniqueness of the Solution

The general form of a first-order ordinary differential equation (ODE) is given by:

$z' = g\,(t, z),\ z\,(t_o) = z_o$ (8)

Utilizing the following theorem, we can affirm the existence and uniqueness of the solution for the model under consideration.

Theorem 1. (Uniqueness of Solution): Define the domain D as:

$|t-t_o| \leq a,\ |z-t_o| \leq b,\ z = (z_1, z_2, \ldots, z_n),\ z_o = (z_{10}, z_{20}, \ldots, z_{no}),$ (9)

and assume that the function $h\,(t, z)$ satisfies the Lipschitz condition:

$|h\,(t, z_1) - h\,(t, z_2)| \leq c\,|z_1-z_2|,$ (10)

for any pairs (t, z_1) and (t, z_2) within D, where c is a positive constant. It follows that there exists a constant $\delta > 0$ such that within the interval $|t-t_o| \leq \delta$, a unique, continuous vector solution $z(t)$ exists for the system. That is, the condition is met when the partial derivatives $\frac{\partial h_i}{\partial z_j}$ for $i, j = 1, 2, \ldots, n$ are continuous and bounded within D.

Lemma 1. If the continuous partial derivatives of $h\,(t, z)$, that is $\left(\frac{\partial h_i}{\partial z_j}\right)$ exist for a closed, convex, and bounded domain, then these derivatives satisfy the Lipschitz condition in the given domain. We focus on the domain: $1 \leq t \leq T$. Thus, we seek a solution in line with the condition, $0 < t < \infty$.

With these foundations, the existence theorem can now be demonstrated effectively.

Theorem 2. *Assume D represents the domain of (6) in such a manner that (7) and (8) hold. Then the bounded solution in domain D of (6) exists.*

Proof.

Let model equations in (6) be represented as;

$$\left.\begin{aligned}
h_1 &= \Pi - \frac{\beta_1 E_c E_x}{N_a} - \frac{\beta_2 E_c R_e}{N_a} - \frac{\beta_4 E_c F_d}{1+\alpha F_d} - \frac{\beta_5 E_c S_c}{N_a} - (\mu + \rho_1)E_c,\\
h_2 &= \rho_1 E_c + \sigma_1 \frac{\beta_5 E_c E_x}{N_a} - \sigma_2 \frac{\beta_3 E_x S_c}{N_a} - \frac{\beta_6 E_x R_e}{N_a} - (\delta_o + \mu)E_x,\\
h_3 &= \frac{\beta_2 E_c R_e}{N_a} + \frac{(1-\eta_1)\beta_1 E_c E_x}{N_a} + \frac{(1-\eta_2)\beta_4 E_c F_d}{N_a} - (\delta_1 + \mu)R_e,\\
h_4 &= \frac{(\eta_1)\beta_1 E_c E_x}{N_a} + \frac{(\eta_2)\beta_4 E_c F_d}{1+\alpha F_d} - \frac{\beta_3 E_x S_c}{N_a} - \frac{(1-\sigma_1)\beta_5 E_c S_c}{N_a} - \mu S_c,\\
h_5 &= r\left(1 - \frac{F_d}{\kappa}\right)F_d - \delta_3 F_d - -(\theta_1 E_c + \theta_2 E_x + \theta_3 E_c + R_e)F_d
\end{aligned}\right\} \tag{11}$$

Then the partial derivatives of h_1 in (11) become

$$\left.\begin{aligned}
\left|\frac{\partial h_1}{\partial E_c}\right| &= \left|\begin{matrix} -\frac{\beta_1 E_x}{N_a} - \frac{\beta_2 R_e}{N_a} - \\ \frac{\beta_4 F_d}{1+\alpha F_d} - (\mu + \rho_1)\end{matrix}\right| < \infty,\\
\left|\frac{\partial h_1}{\partial E_x}\right| &= \left|-\frac{\beta_1 E_c}{N_a}\right| < \infty,\\
\left|\frac{\partial h_1}{\partial R_e}\right| &= \left|-\frac{\beta_2 E_c}{N_a}\right| < \infty,\\
\left|\frac{\partial h_1}{\partial S_c}\right| &= \left|-\frac{\beta_5 E_c}{N_a}\right| < \infty,\\
\left|\frac{\partial h_1}{\partial F_d}\right| &= \left|-\frac{\beta_4 E_c F_d}{(1+\alpha F_d)^2}\right| < \infty.
\end{aligned}\right\} \tag{12}$$

If the similar solution in (12) is applied to $h_2 - h_5$ in (11), then the bounded solutions are less than infinity. It is therefore concluded that all the partial derivatives of (11) are bounded in *D,* are continuous. Hence from Theorem 1, there exist a unique solution of model in *D*.

Positivity and Boundedness of the Invariant Region of the Model.

It is evident that the model (6), with non-negative initial conditions specified, guarantees a unique solution. We now proceed to demonstrate that all solutions remain non-negative for all time *t* in the interval [0, 1) regardless of the chosen initial conditions.

The ensuing theorem establishes the non-negativity and boundedness of the state variables of (6)

Theorem 3. *For the model (6), the solutions* $E_c(t), E_x(t), R_e(t), S_c(t),$ *with non-negative initial conditions* $E_c(0), E_x(0), R_e(0), S_c(0),$ *will remain non-negative for all time* $t > 0$ *within a positively invariant closed set*

$\Omega = \{(E_c, E_x, R_e, S_c)^T \in \mathbb{R}^{+5} |\ 0 \leq E_c(t), E_x(t), R_e(t), S_c(t) \leq 1\}.$ (13)

Proof. We start by assuming that the initial conditions for system (6) are non-negative. Suppose $t_1 > 0$ is the earliest time at which at least one component reaches zero while the other components remain non-negative over the interval $[0, t_1)$. Next, we will establish that it is impossible for any component to be zero at t_1. Initially, consider $E_c(t_1) = 0$ while other components stay non-negative from $[0, t_1)$. Now, E_c in (6) can be written at t_1 as:

$\frac{dE_c}{dt}|_{t=t_1} = \mu + \Pi > 0,$ (14)
which means that $E_c(t)$ is strictly monotonically increasing at t_1, that is, $E_c(t) < E_c(t_1)$ for all $t \in (t_1 - \epsilon,\ t_1)$ where $\epsilon > 0$. This results to a contradiction, therefore $E_c(t)$ cannot be zero at t_1.
Again, we assume $E_x(t_1) = 0$ and other components are non-negative, then
$\frac{dE_x}{dt}|_{t=t_1} = \beta_3 S_c(t_1) > 0.$ (15)
This means that $E_x(t)$ is strictly monotonically increasing at t_1. Hence there is a contradiction
Next for $R_e(t_1) = 0$ and other components are non-negative
$\frac{dR_e}{dt}|_{t=t_1} = \frac{\beta_2 E_c R_e}{N_a} + \frac{(1-\eta_1)\beta_1 E_c E_x}{N_a} + \frac{(1-\eta_2)\beta_4 E_c F_d}{N_a} \geq 0$ (16)
This also means that $R_e(t)$ is strictly monotonically increasing at t_1. Hence there is a contradiction
In view of the analysis in (13) – (16), the non-human compartment of funding $F_d(t)$ cannot be zero at t_1. We conclude that at such a point t_1 at which at least at least one component is zero does not exist. Hence, all the model components remain non-negative for all time *t > 0.*
Also, to show that the model is positively invariant, the addition of the total human researchers host population in (6) becomes
$\frac{dN_a}{dt} = \Pi - \mu\, N_a(t)\Pi - \mu\, N_a(t),$ (17)
so that
$\frac{dN_a}{dt} + \mu\, N_a(t) \leq \Pi.$ (18)
By the use of the integrating factor technique, $e^{\int \mu dt} = e^{\mu t}$ and applying, $e^{\mu t}$ both sides of (18), one obtains
$e^{\mu t}\frac{dN_a}{dt} + e^{\mu t}\mu\, N_a(t) \leq e^{\mu t}\Pi,$ (19)
so that the integral of (19) becomes
$e^{\mu t}\, N_a(t) \leq \frac{\Pi}{\mu} e^{\mu t} + k,$ (20)
where k is a constant, and
$N_a(t) \leq \frac{\Pi}{\mu} + k e^{-\mu t}.$ (21)
At t = 0,
$k = N_a(0) - \frac{\Pi}{\mu}.$ (22)
Then
$N_a(t) \leq N_a(0) e^{\mu t} + \frac{\Pi}{\mu}(1 - e^{\mu t}).$ (23)
As $t \to \infty$, the human population size approaches $\frac{\Pi}{\mu}$ from below, that is, $N_a(t) \leq \frac{\Pi}{\mu}$.
This implies that $0 \leq N_a(t) \leq \frac{\Pi}{\mu}$ and the model (6) trajectories are bounded so that the feasible solutions of (6) begins and stays in
$\Omega = \{(E_c, E_x, R_e, S_c)^T \in \mathbb{R}^{+5} |\ 0 \leq E_c(t), E_x(t), R_e(t), S_c(t) \leq 1\}.$ (24)
Hence, Ω is positively invariant and it is sufficient to consider the dynamics of the flow induced by (6).

Stability Analysis of the Model

Stability analysis of the model system (6) is carried out to determine how the model will behave in the long term. The asymptotic stability, where the state of the system converges to an equilibrium point as time goes to infinity, is particularly useful for understanding whether the system will settle into a steady state and what that state will be in future. The Theorems and Lemmas below are supported by the works of (La-Salle 1960, 1962, 1966, 1967, 1968) and (La-Salle and Lefschetz 1961).

Lemma 1. (Local stability) Let $x = 0$ be an equilibrium point for the differential equation $\dot{x} = f(x)$, where $f: R^n \to R^n$ is continuously differentiable. If all eigenvalues of the Jacobian matrix $Df(0)$ have negative real parts, then the equilibrium point $x = 0$ is locally stable.

Lemma 2. Under the conditions of Lemma 2, if all eigenvalues of the Jacobian matrix $Df(0)$ have strictly negative real parts, then the equilibrium point $x = 0$ is locally asymptotically stable.

Types of equilibrium solutions possible in such compartment models: Because all state variables are nonnegative, any equilibrium solution will fall into one of these categories:

- Trivial equilibrium: all researcher compartments are zero, possibly with F_d at some logistic equilibrium value (e.g., if $S_c = 0$, then dF_d/dt reduces to a logistic with depletion).
- Funding-free equilibrium: research populations may exist, but $F_d = 0$.
- Success-free equilibrium: $S_c = 0$ but F_d may be positive or zero.
- Full positive equilibrium: all compartments E_c, E_x, R_e, S_c, F_d are strictly positive.
- Degeneracy due to recruitment Π:

If Π > 0, a fully trivial equilibrium ($E_c = E_x = R_e = S_c = 0$) is not possible because $\frac{dE_c}{dt} = \Pi > 0$ would push E_c upward. This means that some class of researchers will always be present, so equilibrium states will necessarily have $E_c > 0$.

- Equilibrium solving process (qualitative):
  - From the funding equation: This yields either:
    - $F_d = 0$, with S_c satisfying the rest of the system; or
    - $F_d > 0$, determined uniquely in terms of S_c.

- Substituting F_d back into the researcher equations reduced the problem to 4 coupled equations in E_c, E_x, R_e, S_c.
- These can have:
  - One funding-free researcher equilibrium $F_d = 0$,
  - One or more positive funding equilibria where $F_d > 0$ and S_c positive.

Theorem 4. Let H (z)=P (z, ex) where P (z, w) is a polynomial in z and w with the highest order term in zz given as the principal term. Consider the function $H\ (iy)$ for $y \in R$, which can be expressed in terms of its real and imaginary components as: $H\ (iy) = (y) + iG\ (y)$

Statement 1: If all zeros of (z) lie in the left half-plane, then the zeros of $F(y)$ and $G(y)$ are real, simple, and alternate. Moreover, for every $y \in R$, the following inequality holds: (0) $G'(0) - F'(0)\ G(0) > 0$.

Converse Statement: The zeros of (z) will all be in the left half-plane if any of the following conditions are met:

- All zeros of (y) and $G\ (y)$ are real, simple, and alternate, and the inequality $F(y)\ G'(y)-F'(y)\ G(y)>0$ is satisfied for at least one yy.
- All zeros of (y) are real and for each zero $y0$ of F, the inequality $F(y0)\ G'(y0) - F\ '(y0)\ G(y0)>0$ is satisfied.
- All zeros of (y) are real and for each zero $y0$ of G, the inequality $F(y0)\ G'\ (y0) - F'\ (y0)\ G(y0) > 0$ is satisfied.

Theorem 5. *The model (6) academic career development - present equilibrium solution*

$$\mathbb{U}^* = (E_c^*, E_x^*, R_e^*, S_c^*, F_d^*) = \left\{\begin{array}{l}
E_c^* = \dfrac{\Pi}{\frac{\beta_1 E_x^*}{N_a}+\frac{\beta_2 R_e^*}{N_a}+\frac{\beta_4 E_c F_d}{1+\alpha F_d}+\frac{\beta_5 S_c^*}{N_a}+\mu+\rho_1}, \\
E_a^* = \left(\dfrac{\rho_{1E_c^*}}{(\delta_o+\mu)+\frac{(\sigma_2)\beta_3 E_c^*}{N_a}+\frac{\beta_6 R_e^*}{N_a}-\frac{(\sigma_1)\beta_5 E_c^*}{N_a}}\right), \\
S_c^* = \left(\dfrac{\frac{(1-\eta_1)\beta_1 E_c^* E_a^*}{N_a}-\frac{\beta_4 E_c F_d}{1+\alpha F_d}}{\mu+\frac{\beta_3 E_x^*}{N_a}+\frac{(1-\sigma_1)\beta_5 E_c^*}{N_a}}\right), \\
R_e^* = \dfrac{1}{N_a}\left(\dfrac{\frac{\eta_1\beta_1 E_c^* E_x^*}{N_a}+\frac{\beta_4 E_c F_d}{1+\alpha F_d}}{(\delta_1+\mu)-\frac{\beta_3 E_x^*}{N_a}}\right), \quad \text{where } (\delta_1+\mu) > \dfrac{\beta_3 E_x^*}{N_a}, \\
F_d^* = 0 \ or\ F_d^* = k\left(\frac{r-\delta_3}{r}\right) if\ r > \delta_3, \\
0 = r\left(1-\frac{F_d}{\kappa}\right)F_d - \delta_3 F_d + \alpha_s S_{c.} \Leftrightarrow \kappa r {F_d}^2 + (r-\delta_3)F_d + \alpha_s S_c. \\
\text{For a given } S_c \geq 0, \text{ this quadratic yields,} \\
{F_d}^* = \frac{k}{2}(1-\frac{\delta_3}{r}) \pm \sqrt{(1-\frac{\delta_3}{r})^2 + \frac{4\alpha_s}{rk} S_c^*},
\end{array}\right. \tag{25}$$

is locally asymptotically stable.

Proof.

The model is linearized around the equilibrium solutions $\mathbb{U}^*$ in (25), by obtaining its Jacobian matrix $\mathbb{J}$ given

$$\mathbb{J}(\mathbb{U}^*) = \begin{pmatrix} -a^* & -\vartheta_1^* & -\vartheta_2^* & 0 & \vartheta_3^* \\ 0 & -b^* & 0 & -\vartheta_6^* & 0 \\ -\vartheta_1^* & \vartheta_2^* & -c^* & 0 & 0 \\ \vartheta_4^* & \vartheta_5^* & 0 & -d^* & \vartheta_7^* \\ 0 & 0 & 0 & 0 & -e^* \end{pmatrix}, \tag{26}$$

Where

$$\left.\begin{array}{l}
a^* = -\dfrac{(1+\eta_1)\beta_1 E_x^* - \beta_2 R_e^* - (1+\eta_2)\beta_4 F_d^* - \mu}{N_a}, \\
\vartheta_1^* = \dfrac{(1+\eta_2)\beta_1 E_c^*}{N_a}, \\
\vartheta_2^* = \dfrac{\beta_2 E_c^*}{N_a}, \\
\vartheta_3^* = \dfrac{\beta_4 E_c^*}{N_a}, \\
b^* = \dfrac{-(\delta_o+\mu)\beta_3 S_c^*}{N_a}, \\
\vartheta_4^* = \dfrac{\beta_1 E_x^* + \beta_4 F_d^*}{N_a}, \\
\vartheta_5^* = \dfrac{\beta_1 E_c^* - \beta_3 S_c^*}{N_a}, \\
\vartheta_6^* = \dfrac{-\beta_3 S_c^*}{N_a}, \\
c^* = -\dfrac{(\delta_1+\mu)\beta_2 E_c^*}{N_a}, \\
d^* = -\mu, \\
e^* = \dfrac{-\delta_3 + r(2-r)}{k}.
\end{array}\right\} \tag{27}$$

The characteristics polynomial of (27) is given by

$\lambda^5-(-e^*-d^*-c^*-b^*-a^*)\lambda^4-(-e^*d^*-e^*c^*-e^*b^*-e^*a^*-c^*d^*-b^*d^*-a^*d^*-c^*b^*-c^*a^*-b^*a^*+\vartheta_1^*\vartheta_2^*-\vartheta_5^*\vartheta_6^*)\lambda^3-(-e^*d^*c^*-e^*d^*b^*-e^*d^*a^*-e^*c^*b^*-e^*c^*a^*-e^*b^*a^*+e^*\vartheta_1^*\vartheta_2^*-e^*\vartheta_5^*\vartheta_6^*-d^*c^*b^*-d^*c^*a^*-d^*b^*a^*+e^*\vartheta_1^*\vartheta_2^*-c^*b^*a^*-\vartheta_5^*\vartheta_6^*c^*+\vartheta_1^*\vartheta_2^*b^*-\vartheta_5^*\vartheta_6^*a^*+\vartheta_1^*\vartheta_2^*\vartheta_6^*)\lambda^2-(-e^*d^*c^*b^*-e^*d^*c^*a^*-e^*d^*c^*a^*+e^*d^*\vartheta_1^*\vartheta_2^*-e^*c^*b^*a^*-e^*c^*\vartheta_5^*\vartheta_6^*+e^*b^*\vartheta_1^*\vartheta_2^*-e^*a^*\vartheta_5^*\vartheta_6^*+e^*\vartheta_1^*\vartheta_2^*\vartheta_6^*-d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*+c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*+\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*+\upsilon_2^2\vartheta_4^*\vartheta_6^*)\lambda+e^*(d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*-c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*-\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*-\upsilon_2^2\vartheta_4^*\vartheta_6^*).$ (28)

In view of Theorem 4, the real and imaginary parts are separated to that replacing λ with $i\lambda$, and $p(\lambda)=F(\lambda)+iG(\lambda)$ one obtains,

$F(\lambda)=-(-e^*-d^*-c^*-b^*-a^*)\lambda^4+(-e^*d^*c^*-e^*d^*b^*-e^*d^*a^*-e^*c^*b^*-e^*c^*a^*-e^*b^*a^*+e^*\vartheta_1^*\vartheta_2^*-e^*\vartheta_5^*\vartheta_6^*-d^*c^*b^*-d^*c^*a^*-d^*b^*a^*+e^*\vartheta_1^*\vartheta_2^*-c^*b^*a^*-\vartheta_5^*\vartheta_6^*c^*+\vartheta_1^*\vartheta_2^*b^*-\vartheta_5^*\vartheta_6^*a^*+\vartheta_1^*\vartheta_2^*\vartheta_6^*)\lambda^2+e^*(d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*-c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*-\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*-\upsilon_2^2\vartheta_4^*\vartheta_6^*),$ (29)

and

$G(\lambda)=i[\lambda^5+-(-e^*d^*-e^*c^*-e^*b^*-e^*a^*-c^*d^*-b^*d^*-a^*d^*-c^*b^*-c^*a^*-b^*a^*+\vartheta_1^*\vartheta_2^*-\vartheta_5^*\vartheta_6^*)\lambda^3-(-e^*d^*c^*b^*-e^*d^*c^*a^*-e^*d^*c^*a^*+e^*d^*\vartheta_1^*\vartheta_2^*-e^*c^*b^*a^*-e^*c^*\vartheta_5^*\vartheta_6^*+e^*b^*\vartheta_1^*\vartheta_2^*-e^*a^*\vartheta_5^*\vartheta_6^*+e^*\vartheta_1^*\vartheta_2^*\vartheta_6^*-d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*+c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*+\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*+\upsilon_2^2\vartheta_4^*\vartheta_6^*)\lambda.$ (30)

Obtaining the derivatives of both the real and imaginary parts with respect to λ so that

$F'(\lambda)=-4(-e^*-d^*-c^*-b^*-a^*)\lambda^3+2(-e^*d^*c^*-e^*d^*b^*-e^*d^*a^*-e^*c^*b^*-e^*c^*a^*-e^*b^*a^*+e^*\vartheta_1^*\vartheta_2^*-e^*\vartheta_5^*\vartheta_6^*-d^*c^*b^*-d^*c^*a^*-d^*b^*a^*+e^*\vartheta_1^*\vartheta_2^*-c^*b^*a^*-\vartheta_5^*\vartheta_6^*c^*+\vartheta_1^*\vartheta_2^*b^*-\vartheta_5^*\vartheta_6^*a^*+\vartheta_1^*\vartheta_2^*\vartheta_6^*)\lambda,$ (31)

and

$G'(\lambda)=i[5\lambda^4+-(-e^*d^*-e^*c^*-e^*b^*-e^*a^*-c^*d^*-b^*d^*-a^*d^*-c^*b^*-c^*a^*-b^*a^*+\vartheta_1^*\vartheta_2^*-\vartheta_5^*\vartheta_6^*)3\lambda^2-(-e^*d^*c^*b^*-e^*d^*c^*a^*-e^*d^*c^*a^*+e^*d^*\vartheta_1^*\vartheta_2^*-e^*c^*b^*a^*-e^*c^*\vartheta_5^*\vartheta_6^*+e^*b^*\vartheta_1^*\vartheta_2^*-e^*a^*\vartheta_5^*\vartheta_6^*+e^*\vartheta_1^*\vartheta_2^*\vartheta_6^*-d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*+c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*+\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*+\upsilon_2^2\vartheta_4^*\vartheta_6^*).$ (32)

Setting $\lambda=0$ in (31) and (32), that is, $F(0), G(0), F'(0)$ *and* $G'(0)$, the expressions in (31) – (32) reduces to

$F(0)=e^*(d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*-c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*-\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*-\upsilon_2^2\vartheta_4^*\vartheta_6^*,$ (33)

$G(0)=0,$ (34)

$F'(0)=0,$ (35)

$G'(0)=(-e^*d^*c^*b^*-e^*d^*c^*a^*-e^*d^*c^*a^*+e^*d^*\vartheta_1^*\vartheta_2^*-e^*c^*b^*a^*-e^*c^*\vartheta_5^*\vartheta_6^*+e^*b^*\vartheta_1^*\vartheta_2^*-e^*a^*\vartheta_5^*\vartheta_6^*+e^*\vartheta_1^*\vartheta_2^*\vartheta_6^*-d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*+c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*+\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*+\upsilon_2^2\vartheta_4^*\vartheta_6^*).$ (36)

By the use of the statement in Theorem 6,

$F(0)G'(0)-F'(0)G(0)=e^*(d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*-c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*-\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*-\upsilon_2^2\vartheta_4^*\vartheta_6^*)(-e^*d^*c^*b^*-e^*d^*c^*a^*-e^*d^*c^*a^*+e^*d^*\vartheta_1^*\vartheta_2^*-e^*c^*b^*a^*-e^*c^*\vartheta_5^*\vartheta_6^*+e^*b^*\vartheta_1^*\vartheta_2^*-e^*a^*\vartheta_5^*\vartheta_6^*+e^*\vartheta_1^*\vartheta_2^*\vartheta_6^*-d^*c^*b^*a^*-d^*b^*\vartheta_1^*\vartheta_2^*+c^*a^*\vartheta_5^*\vartheta_6^*+c^*\vartheta_1^*\vartheta_4^*\vartheta_6^*+\vartheta_1^*\vartheta_2^*\vartheta_5^*\vartheta_6^*+\upsilon_2^2\vartheta_4^*\vartheta_6^*)$ (37)

Since we have shown that $F(0)G'(0)-F'(0)G(0)>0$, then we conclude that the model academic career development present equilibrium is locally asymptotically stable.

Numerical Results

Here, the numerical simulations are carried out via the RK4 embedded numerical scheme in python computational software (Chowdury *et al.* 2019). We assume initial population of academia in academic host community described by the variables $E_c=50, E_x=30, R_e=20, S_c=10, F_d=50$, while the parameter values used for the purpose of model simulations are drawn from the works of (Mazzoleni et al. 2021) given by $\Pi=1.0, \eta_1=0.1, \eta_2=0.12, \beta_1=0.05, \beta_2=0.03, \beta_3=0.04, \beta_4=0.06, \mu=0.002, \delta_o=0.4, \delta_1=0.24, r=0.7, \kappa=100, \delta_3=0.01$.

In Figure 1, the asymptotic behavior of the model at the academic career development free equilibrium solution starts with small initial values that ultimately converges to a stable equilibrium as time progresses. The fast convergence suggests that the academic career development free equilibrium is stable, and the system tends to return to this state even with small disturbances. This shows that policymakers in academia need to respond quickly to small disturbances due to problems encountered by early career researchers.

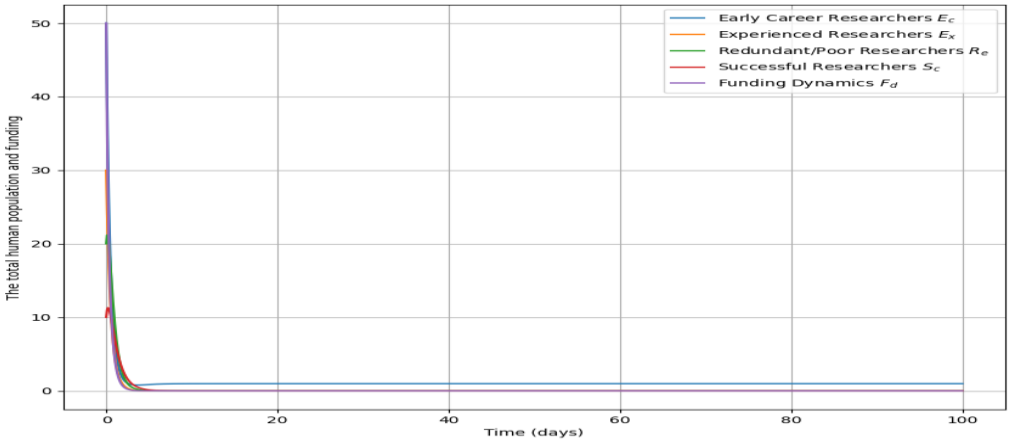

Figure 1: The behaviour of the asymptotic stability of the model on academic career-free equilibrium.

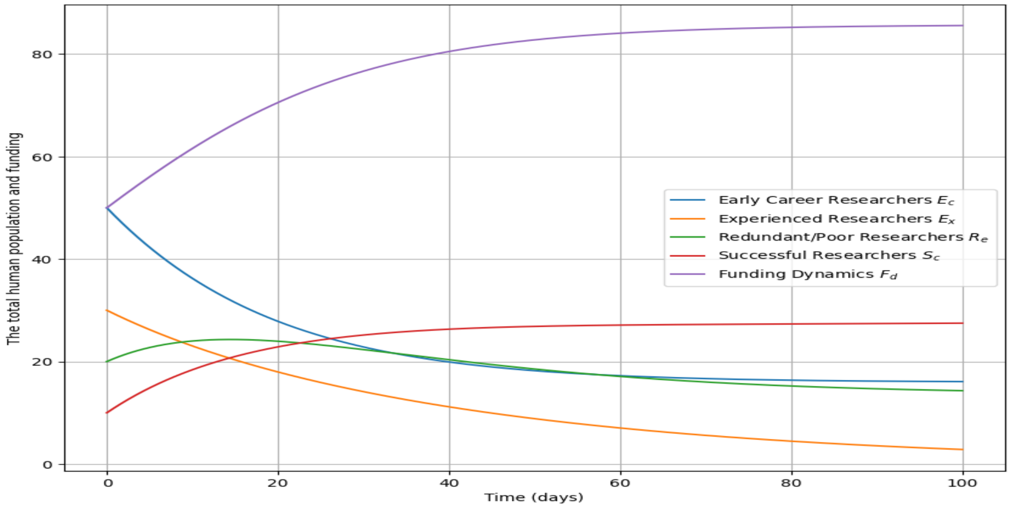

Figure 2: The behaviour of the asymptotic stability of the model variables on academic career-present equilibrium.

Furthermore, in Figure 2, the curve depicting ECRs shows that initially, there is a gradual decrease in the population of ECRs due to recruitment (Π). The population decreases faster over time as ECRs interact with experienced researchers E_x, redundant/poor researchers R_e, and funding dynamics F_d, which results to a decline as they transition to other states. Also, the population of experienced researchers increase initially when the contact/influence rate β_3 with successful researchers S_c is sufficiently high leads to a gradual decline in the population of experienced researchers over time due to probable retirement or natural death. In addition, the curve illustrating the population of redundant/poor researchers increase initially due to interactions with ECRs η_1, experienced researchers η_2, and funding dynamics β_4, but decline due to dropout/failure rate δ_o and natural death μ which contribute to the reduction of the population of redundant/poor researchers over time. Moreover, the curve of the population of successful researchers increase initially due to positive interactions with ECRs β_1 and funding dynamics, β_4. The increase in the population of successful researchers increase and stabilize or decline over time. Finally, the curve of the funding dynamics follows a logistic growth pattern, initially increasing rapidly but eventually reaching a steady state due to the carrying capacity κ and funding depletion δ_3. ;Analyzing these behaviors in Figures 1 and 2, gives the understanding of how various factors interact to shape the trajectories of different researcher categories over time.

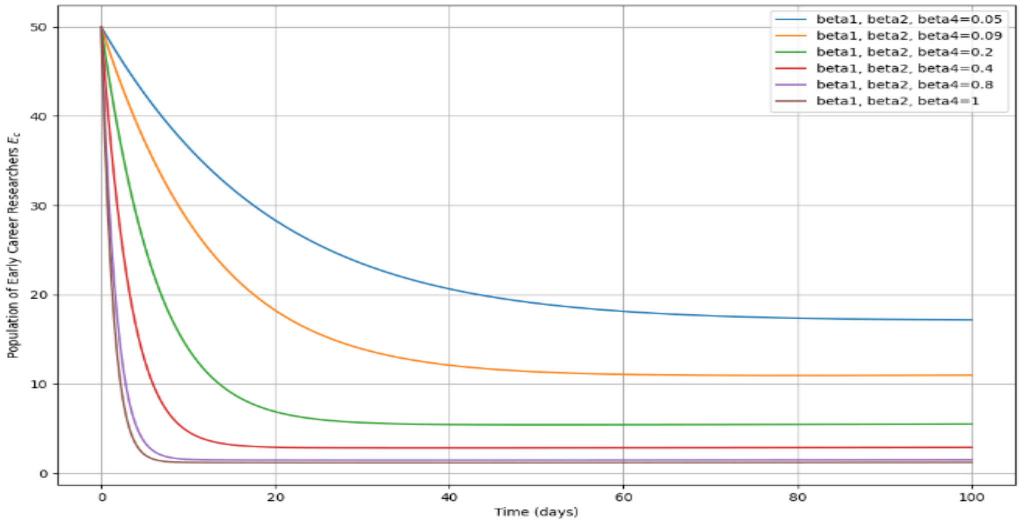

Figure 3: The behaviour of the model variable E_c varying influence rates $\beta_1, \beta_2, \beta_4$ fixing other parameters.

In Figure 3, varying β_1, β_2 and β_4, collectively in each iteration allows us to see the combined impact of these interaction rates on E_c . Each line on the plot represents a different set of interaction strengths. Generally, increasing these interaction rates might either deplete E_c faster (if the interactions leads to transitioning to R_e) or promote growth (when interactions with funding F_d are predominantly beneficial). By examining the plots, one can discern patterns and identify which interaction strength levels are most conducive to sustaining or increasing the population of E_c over time. This graphical behavior provides a clearer picture of how various interaction dynamics influence early career academic trajectories and assists in understanding critical thresholds or optimal interaction rates for fostering a supportive academic environment.

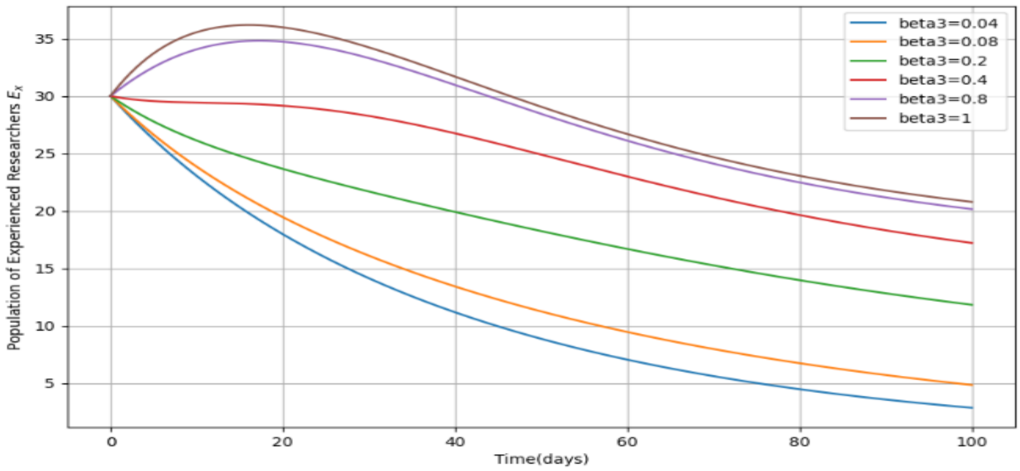

Figure 4: The behaviour of the model variable E_x varying influence rates β_3, fixing other parameters.

In Figure 4, increasing β_3 enhances the interaction strength between E_x and S_c, which could lead to a higher retention or even growth of E_x if these interactions are beneficial (e.g., through mentorship, collaboration on projects). A higher β_3 suggests more effective or frequent positive engagements that could help sustain or elevate the status of E_x . By examining the plots for each value of β_3, one can see how the population dynamics of E_x change. If the plot shows an increase or more stable population with higher β_3, it indicates that the interactions with S_c are likely beneficial. This focused analysis provides insights into the critical role of successful collaborations and mentorship in supporting and enhancing the career paths of experienced researchers within academic environments.

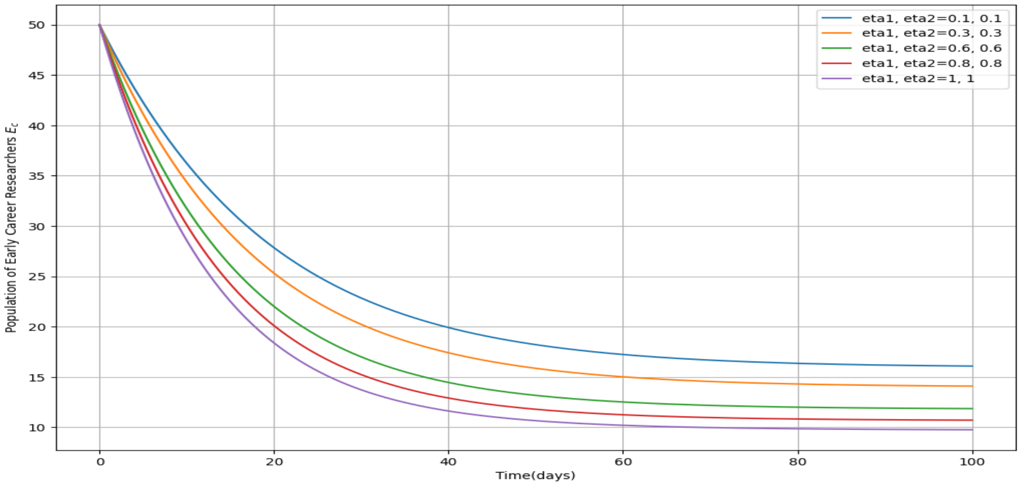

Figure 5. The behaviour of the model variable E_c varying influence rates η_1, and η_2 fixing other parameters constant.

In Figure 5, Increasing η_1, and η_2 indicates a higher proportion of negative outcomes from interactions between E_c and E_x, and between E_c and funding F_d. If these parameters are high, it suggests that the interactions are more detrimental, potentially leading to a faster decline in the population of E_c due to increased transition to less favorable states (like R_e). By examining the plots for each combination of η_1, and η_2, one can see how the dynamics of E_c change as the risk of negative interactions increases. A visible decline in E_c would indicate the critical impact of these factors and highlight the need for improved supportive measures in academic environments.

Discussion

The findings from this study align with and extend previous research on the use of mathematical modeling to describe population-type transitions in professional and social systems cited in the work. We developed nonlinear compartmental models to capture productivity-related and behavioral transitions in academic institutional populations. However, while those studies focused primarily on unidirectional and homogeneous interactions, Mazzoleni *et al*. (2021), the present model introduces bidirectional, funding-mediated, and nonlinear interactions, thus capturing the multi-dimensional feedback between researcher categories and the non-human compartment of funding together with a Holling type II functional response to represent the diminishing marginal effect of funding on research success. This addition corrects for the unrealistic assumption of limitless resource benefits by providing a more accurate reflection of institutional funding constraints and their effect on career advancement. Thus, the inclusion of the Holling II term marks a significant methodological improvement that bridges theoretical modeling and the empirical realities of academic ecosystems.

This work captures the dynamic interplay among Early Career Researchers (E_c), Experienced Researchers (E_x), Redundant/Poor Researchers (R_e), Successful Researchers (S_c), and the Funding pool (F_d). Through positivity and invariance analysis, all state variables were shown to remain nonnegative and bounded within an institutionally feasible region to confirm the social realism of the model. This ensures that trajectories do not escape beyond meaningful states. This is an important feature that reinforces the system's analytical soundness.

In view of the equilibrium solutions, local stability analysis, conducted via the Jacobian matrix, demonstrated that the academic stagnation-free equilibrium is locally and globally asymptotically stable to show transitions to a success prevalence regime in order to reveal that reinforcing mentorship and funding mechanisms sustain long-term productivity.

Figure \1, demonstrates the asymptotic behavior near the career-development–free equilibrium, where trajectories converge to a stable state despite small perturbations. This outcome confirms the system's local asymptotic stability and resilience, as predicted by 's invariance principle. It implies that temporary disruptions such as delayed funding or mentoring setbacks do not permanently destabilize the academic environment.

Figure 2 illustrates the evolution of the career-present equilibrium, where the Early Career Researchers (E_c) compartment initially rises due to recruitment inflow (Π) before stabilizing as members transition to E_x, R_e, or S_c. The observed fluctuations among E_c and E_x correspond to the cyclical nature of academic progression, driven by mentorship, collaboration, and competition. Over time, redundant researchers (R_e) decline as a result of loss ($\delta_1 + \mu$) and institutional reintegration efforts, in line with **(**Boeren et al. 2015, Laudel and Gläser 2008, Bridle et al. 2013**)**, who noted that underperforming academics often exit or adapt within structured support systems. The logistic rise of funding (F_d) toward a saturation point validates findings from Khoo et al. (2023) and Mazzoleni et al. (2021) on the finite and self-regulating nature of academic funding ecosystems.

Figures 4 and 5 reveal the impact of varying interaction parameters on career progression. Adjustments in mentorship (β_1), negative peer influence (β_2), and funding access (β_4) significantly modify the trajectory of Early Career Researchers. Higher mentorship effectiveness enhances E_c growth, while increased negative influence or misallocated funding (η_1, η_2) accelerates decline. These findings reinforce the dualistic nature of mentorship and competition described by Sutherland (2017) and Bridle et al. (2013).

When collaboration between experienced and successful researchers (β_3) increases, the retention of E_x improves, corroborating the empirical evidence that strong mentorship and collaboration promote longevity and productivity (Boeren et al. 2015, Mason and Merga 2022, Sherif et al. 2020). Furthemore, exposure to high redundancy (β_6) undermines overall performance, aligning with Sørensen and Kristensen (2022), who found that negative institutional climates erode research motivation and innovation.

By combining analytical rigor with numerical simulations, this model provides a comprehensive understanding of how academic systems maintain equilibrium under fluctuating recruitment, funding, and mentorship conditions. The integration of nonlinear functional responses and feedback mechanisms advances quantitative modeling in social and institutional dynamics (Mazzoleni et al. 2021, Bansal et al. 2023).

Conclusion

In this study, we developed a comprehensive mathematical model to analyse the dynamics among different categories of researchers within academia, with a particular focus on how variables such as mentorship, funding access, and academic peer influence affect the career trajectories of early career, experienced, redundant, and successful researchers. The model applies stability theory and relevant differential equations theorems to affirm the existence, uniqueness, positivity, and boundedness of the model solutions. Equilibrium solutions were computed to analyse the local asymptotic stability of the model. The numerical simulations, executed using the Runge-Kutta fourth-order method (RK4) and implemented in Python, provided vivid insights into the career growth challenges within the academic community. Finally, the study emphasizes that managing and optimizing interactions within academic settings is crucial for enhancing the success rates of researchers, particularly those in the early stages of their careers. By identifying the thresholds and optimal interaction rates through our model, academic institutions can foster a more supportive and productive environment, thereby enhancing overall academic success and sustainability.

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